# MATHEMATICS TEACHING

THE BULLETIN OF THE
ASSOCIATION FOR
TEACHING AIDS IN
MATHEMATICS

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No. 10—JULY 1959 Price 3/6

MATHEMATICS TEACHING

Published three times yearly

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Contributions to the journal and enquiries for advertising space should be addressed to the Editor. Contributors are asked to keep mathematical notation as typographically simple as they can, and to submit illustrations, where necessary, in a form suitable for reproduction.

Enquiries about distribution should be addressed to the Secretary of the Association.

Coulton & Co. Ltd., Printers, Nelson and Colne.

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# EDITORIAL

## THE NEED FOR A PHILOSOPHY

We live in an age of specialisation. Division of labour is an essential basis of civilised existence; without it our society could not develop, for it enables the individual to give his personal attention to one important aspect of life within society. Thereby knowledge of that aspect is increased, knowledge as a whole is enriched and society takes a step forward. Moreover, as the fund of knowledge is increased, it is no longer possible for each individual to encompass it. He therefore devotes his attention to one particular field, and the process of specialisation is further extended. Consequently each individual becomes a specialist concentrating on his own particular branch of knowledge and activity. He concentrates on learning the special facets of his job, the particular ways of overcoming the difficulties which arise and the most effective way of achieving his desired objectives.

But there is a grave danger in all this: we are tending to over-specialisation. The outcry is that our scientists are not receiving a liberal education (although this argument has been answered in its turn) and that their studies should be widened by adding an ever-increasing number of subjects to the curriculum. Just how many quarts it is hoped to squeeze into a pint of student's timetable is not stated. Of course, the same may be said of the Arts student: is the Arts specialist who has no knowledge of the basic principles of science a person who can be regarded as anything but a narrow specialist in the modern world? And what of the mathematician who so divorces himself from the realities of existence as to live in a world of pure abstraction with a language of its own?

The danger of all this is that one only sees the problems of society and the problems of one's fellows from the standpoint of one's own particular speciality. Under these circumstances the human aspect tends to disappear; we lose a richness and fullness of life and become a shadow of our true selves; we are little more than human automata trained to deal with one particular set of circumstances and useless when transferred to a different environment. The problem may be expressed in many different guises: the call for a liberal education; the need to humanize the individual; the fight against the evils of authoritarianism. But in essence it is the need to combat over-specialisation.

The part that we, as teachers, have to play in this is vital; this, in fact, is our battle. It is we who are the humanizers of society: it is for us to bring to our pupils an appreciation of the fuller life and an awareness of the danger of narrow specialisation. Unfortunately, however, we are becoming increasingly victims of this move to specialisation ourselves. No longer do we regard ourselves as teachers; instead we are 'mathematicians,' 'chemists,' 'mathematics specialists,' 'English specialists,' 'Heads of Departments' and so on through a whole list of pretentious titles. A glance through the vacancy columns of the educational press will bring home this point clearly. But it is not merely the wording of advertisements at which criticism can be levelled; we do really believe that we are the people described by these titles. Any Secondary School Headteacher knows that an advertisement for a teacher of general subjects would produce little or no response. We are all anxious to teach 'our subject'; we are 'mathematicians,' so we must teach that subject.

It might be argued that a person with qualifications in mathematics is better engaged in teaching that subject than any other, and this is admitted. What is being argued is that an attitude of mind is developing where we put our subject first and our work as teachers second—and a poor second at that! We are so concerned with driving home our own particular branch of knowledge that we overlook our prime function as educators. And by so doing we are ignoring our responsibilities.

In an age of specialisation it is the specialist who must create the liberalising influence of the time. It is he who must see his studies and his work in true perspective; it is for him to set them in their environment. He must no longer think of his subject in isolation, but in its relation to other branches of knowledge. This is what distinguishes the teacher from the specialist. The specialist may be concerned with the techniques of his trade, but the true teacher places prime importance on an interpretation of life through his subject. This is the philosophy for which each individual teacher must seek; and having found it, the knowledge which he must impart. A liberal education starts, not with the subjects taught, but with those who teach them; it is not what is taught so much as how it is taught.

This is the challenge of the 'Specialist Age.' Are we, as teachers, prepared to meet it? Have we thought enough about our subject? Do we understand it in all its aspects? Do we fully understand its implications to our day and age? Is our method of presentation adequate; does it stimulate thought not merely in the subject itself but in wider fields? Are we giving our pupils educative experience in its fullest sense? Until we are able to answer these and similar questions in the affirmative we cannot disclaim our description as 'specialists' in the worst sense of the word, nor pretend that we are discharging our responsibilities as the *teachers* of the modern world.

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## A PERSONAL MESSAGE.

One always is wiser after the event. When the previous Editor of *Mathematics Teaching*, Trevor Fletcher, indicated that he wished to relinquish his post in order to devote more time to the production of mathematical films, I allowed myself to be talked into taking over the Editorship. It was only later that I realized what was involved and the enormous debt the Association owes to my predecessor for having established the magazine so firmly.

The first Bulletins of the Association were duplicated sheets, but in November 1955 *Mathematics Teaching* appeared in its present format. Since then it has established itself as a journal which presents to the world all that is best in the teaching of mathematics in this country. Mr. Fletcher has worked hard to achieve this, and it is now my task to continue the work.

As I see it there are two major problems which face us. The first is to establish the magazine on a firm financial basis by increasing its circulation. Secondly, we must aim to maintain the high standard of the magazine. In both these matters I know I have the whole-hearted support of my fellow Committee members of A.T.A.M. But valued as their support is, these ends will be hard to achieve without the assistance of you, our readers. We want articles for future issues and we invite those who are

able to contribute to get in touch with us; we shall be happy to welcome new writers and every assistance will be given to them.

It must be admitted, however, that the number of people capable and willing to supply original material is limited. But the matter of circulation is one that can be tackled by all. The Association frequently receives expressions of appreciation of what it is doing to improve the standard of mathematical teaching. You can assist by bringing more people into the fold. Let us have more people at our meetings, more readers of *Mathematics Teaching* and more members of A.T.A.M. As mathematics teachers the improvement in teaching our subject at all levels is one which will ultimately benefit us all. Here, then, is *your* task: help us to increase our membership *now*.

For our part, we intend not only to maintain the standards already established by this magazine, but to go on to even greater achievements.

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# SECONDARY MODERN SCHOOL MATHEMATICS — I.
## THE TEACHER IN TRAINING.

ALEC D. WALTERS.

(This is the first of a series of articles on aspects of mathematics teaching in Secondary Modern Schools. The most important factor in the success or otherwise of this work is the teacher himself, and in this article Mr. Walters, who is the Senior Mathematics Lecturer at Padgate Training College, Warrington, considers the work being done in the training of mathematics teachers).

In writing about this subject my own experience must necessarily be to the fore, and I shall confine my attention solely to the Training Colleges. On entry to college, students may roughly be divided into three classes as far as attitude to mathematics is concerned. The first group consists of a small but increasing number of men and women who choose to study mathematics as one of their special topics and from whose ranks the specialist teachers will largely be drawn. At the other extreme we have a group, far too large and almost entirely feminine, who fear, and in some cases, loathe the subject and whose ability, even in simple arithmetic, is abysmally poor. Of the remainder quite an appreciable number, whose first love is some other subject, are quite keen to carry on with mathematics, and many subsequently find considerable pleasure in teaching it.

Students are generally divided into Secondary, Junior or Infant groups according to the level at which they are contemplating teaching. A high proportion of those whose first speciality is mathematics are potentially Secondary Teachers. The Secondary group is also not without an appreciable number of those who dislike the subject, although more than a fair share of the latter tend to join the Infant group.

The work which the specialist students are doing is, in general, well beyond the level at which they will be teaching. However, in our own teaching at this level we aim to demonstrate what we consider to be good teaching methods and many opportunities occur to draw their attention to points which can be considered in contexts more appropriate to their own eventual circumstances. Colleges in Institutes which are not rigidly bound by formal examinations are luckier in this respect, for the numbing effect of syllabuses on teaching runs right through the educational system and the Training Colleges are not immune.

There are few colleges now which do not give at least one period a week to 'Basic Mathematics' or 'Mathematics Method,' which all students take. In the case of my own college I am able to have the Secondary students divided into four groups in such a way that the first group contains specialist students and the fourth group those whose ability is slight and whose attitude is generally one of fear. This system is distinctly helpful with such a wide range of ability and aptitude.

With all students, in addition to purely 'method' work, in this basic course we try to encourage wide reading about the subject, some consideration of its history and its connection with the environment and with other subjects of study. We also try to help the very weak students to improve their own basic knowledge, and in the case of my college, voluntary classes are held for this purpose.

Although the different groups will reach different levels in their 'method' work certain general principles run throughout the course. One of these is rational teaching, i.e. building new ideas, in general, on the framework which the child

already accepts and understands in such a way that he will accept the new as reasonable and not as a piece of magic which he must simply learn and memorise. Another is the use of practical experience and realistic problem material throughout the teaching of the subject, frequently with the 'sandwich' effect of starting and finishing with experimental and problem material, with formal practice in the middle. Again, as a further example, I consider it valuable to stress the fact of interconnection between subjects; by all means teach geography in the mathematics lesson and vice versa, and do not forget that every teacher of mathematics is a teacher of English.

Particularly at the present time when the shortage of teachers is so acute, we do not assume that our specialist mathematicians will necessarily go into Secondary Modern Schools: they may, and do, find their first teaching post in any of an increasing variety of Secondary Schools, sometimes even in a Grammar School of traditional pattern. This, however, makes little or no difference as far as method training goes, for I am sure that much Selective School mathematics teaching for all but the best forms would benefit by the application of techniques which are in use in the best Secondary Modern Schools. With the specialist group we aim to give a quick survey of Infant and Junior method and the teaching of ordinary arithmetical techniques at Secondary level and then spend an appreciable time discussing the wider fields of mathematics at Secondary level, geometry, trigonometry, algebra, social and civic arithmetic, etc. I feel that in some of these subjects, particularly algebra, a good deal of guidance is necessary in teaching the very important elementary stages if the Modern Schools are not going to copy the syllabus, techniques and speed which are only appropriate to the better forms of the Grammar School.

With the middle group it is not appropriate to go so far with specialist teaching method. The assumption here is that these students will, generally speaking, teach the lower forms, and teach mainly arithmetic, although we aim to cover more than that. Fairly frequently from within this group we receive requests to learn more mathematics, a demand which should of course be encouraged but which, except on a basis of private study, is rather difficult to satisfy within the framework as it now stands.

To the best of my knowledge secondary school head teachers expect all their assistants to be capable of teaching elementary arithmetic, although perhaps in the near future specialisation may lead to a change in this attitude. However, at present we have to do something with our weakest group to try to lead them to conquer their fears, master elementary arithmetic and gain confidence in their ability to teach it. This is not easy and we try a variety of approaches, e.g. remedial work, showing the subject in the widest possible context, appealing to their professional pride and their duty to the children. Some of them ultimately find unexpected pleasure in teaching a subject which they have feared throughout their own school lives. But others, I am afraid, never do.

Most of my students are also given the opportunity of developing some aspect of the subject of their own choosing. This might be a study of the relevance of mathematics to another subject in which they are interested, the reviewing of a series of text-books, studying a new topic, etc.

Visits from practising teachers in secondary schools are often arranged for my specialist group and I have found that these are particularly valuable: students welcome the opportunity to listen to an experienced teacher of mathematics and dicuss problems with him or her, and I should be very pleased to hear from teachers who might wish to help in this way.

Students' experiences on Teaching Practice vary tremendously. All should have the opportunity, particularly on their first practice, of seeing mathematics well taught. Equally important, all should be given a specific topic or topics to teach, particularly on their later practices. It is very frustrating to be told to 'revise multiplication of decimals' with a class who are tired of revising it or, worse still, 'They have got to page 44, carry on from there.' I know that it is much easier in subjects like science, history and geography for schools to give a student a specific piece of work which has not been covered, but I would appeal to teachers of mathematics to give careful thought to providing suitable topics for teaching and also ask them to put the student into the picture as regards the mathematical attainments and abilities of the children. But please do not ask one of our arithmetically illiterate to teach simultaneous equations to a G.C.E. stream—it isn't fair to the children! Some consideration must also be given to the *student's* attainment and ability.

As time goes on I envisage closer co-operation between schools and colleges in the training of teachers, and I welcome this tendency. One result of the introduction of the three-year course is likely to be that one period of teaching practice will be considerably extended and should be more than proportionally worth while. During this time it is hoped that the student's status will be more like that of an extra member of staff and that the school will take a greater share of responsibility for his or her training.

In planning for the three-year course the colleges were pulled different ways by the conflicting interests of different parties, but as the air begins to clear two principles which seemed to have gained general acceptance emerge. It is felt strongly that academic standards must improve and to this end students will be encouraged to study one subject to a high level. There is also the feeling that more time must be given to private study, particularly in the final year when the student is more mature and can reflect more deeply on such matters as educational theory and its relationship to practical teaching. At present we find that so many of our students are conditioned by classroom teaching and working for examinations that it is exceedingly difficult to wean them from this attitude: furthermore it is true of mathematics in particular that the academic level of many of the specialist students is such that they need much detailed help. We hope that the extra year will give the necessary time to develop the habit of independent study to a really satisfactory level.

A further hope is that the academic standard of the students on entry will rise, but this may not be realised. If it is not, then it will be virtually impossible to raise the academic standard of the course, for we feel that in many cases at present the students have reached the limit of their ability. If our standards are raised in spite of the lack of improvement in the standard of entrants, there would be the more disastrous effect that the number of mathematics teachers produced would fall far short of the number required. In mathematics we have been particularly

affected by the comparatively poor standard at entry of many of our specialist students, but my own opinion is that the tide has turned and that we are gradually beginning to get an increasing number of better qualified students; many more are needed if the future demands are to be met for specialist teachers of mathematics in the Modern Schools. It is worth while emphasising that, in girls' schools in particular, any promising pupil who has the desire to study mathematics at Sixth Form level should be given the opportunity to do so; in some cases the answer may be for mathematics to be taken on the Arts side. Having taught for some years in a Grammar School it is with some trepidation that I suggest that there might be a little time made available in the Sixth Form to encourage the habit of independent study and wide reading. The Universities as well as the Training Colleges would welcome any improvement of attitude in this direction.

But there will always be a small number of really good students and this number may well increase. We have been looking beyond the immediate future for these students and most of the Area Training Organisations and other responsible bodies are considering the question of some sort of higher qualification for them. For example, the Training College course for the good mathematician may be allowed to rank as the first part of a University course, thus enabling him to do a short further course at a University and obtain a degree. Another suggestion is that there should be a Diploma in Mathematics accepted on a national basis for which the good student should be able to qualify. There have been no final decisions on these matters, but every effort will be made to bridge the gap between the existing Teacher's Certificate and the degree for those students who have real mathematical ability.

Summing up this particular section, I would say that although the increasing number of places in colleges may help to solve the over-all shortage of teachers and the three-year course help us to turn out more mature and better teachers, the greatest single factor which would produce better mathematics specialists for the secondary schools would be a spectacular improvement in the quality of students on entry.

Considering next the students not specialising in mathematics I would say that there is some sign of improvement in the truly appalling state of mathematical illiteracy of many of the girls who enter college, but there is still far to go. Even if they have come straight from school they have frequently done little or no mathematics for three or four years, and their Grammar School experiences have been discouraging. It is perhaps to be regretted, in this context, that many of the examining boards no longer make provision for the separate subject 'Arithmetic' at Ordinary level. Accepting the situation as it stands, however, I feel that much could be done for the girl who has dropped mathematics in the fourth or fifth forms. Assuming that she is specialising in other subjects in the Sixth and has made up her mind to teach, an imaginative approach to some mathematical topic at her own level would perhaps show the subject in a more attractive light, and at the same time help her to improve her own arithmetic. It might be possible to co-ordinate this study with one being taken to Advanced level or it might be an independent project: for instance, a piece of statistical work could be adapted to either of these. Whatever form this Sixth Form work takes, it is essential that the student must realise that she will be expected to teach a little elementary arithmetic in her chosen profession, and that every effort should be made to ensure that she enters college

with less of the feeling of inadequancy and dislike for the subject which is so prevalent at present. There is a truly vicious circle here and attempts must be made to break it at all possible points.

The question remains as to whether we make the best use of the human material that enters our colleges. Perhaps our greatest difficulty as mathematics lecturers is that we are too thinly spread and have far from sufficient time to do all we want to do. This is a difficulty shared by virtually all mathematics teachers at the present time, but it is accentuated in Training Colleges because most of us must be prepared both to teach our subject at a high level and to give help and advice on the teaching of it at all levels. It is the rare paragon who is such an all-round expert, and one of our hopes of the three-year course is that the lecturers, as well as the students, may have more time for study! The recruitment of additional lecturers is, of course, a further problem.

Our aim is to produce a greater number of teachers, and teachers who have a real understanding of mathematics, an appreciation of its history and the part it plays in modern culture, a liking for teaching the subject, and the ability to do so. We are well aware that our aim is far from being achieved. In this article I have stated some of our difficulties, and indicated the ways in which I feel schools might help. In conclusion, may I say that I shall be happy to receive suggestions, through this periodical or otherwise, as to how we mignt improve our own approach.

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## SOME OBSERVATIONS ON THE LEARNING OF GEOMETRY.

DAPHNE ROSWELL-HARRIS.

### INTRODUCTION.

These observations were made whilst teaching mathematics in a temporary post for one year in a Secondary Commercial School of 350 girls. Entrance was by means of a selection procedure of the kind commonly used nowadays at the age of 11. Unfortunately, it has not been possible to ascertain the I.Q. range of the pupils, owing to the rules of the Education Committee concerned, but it is known that the range is small, that the average I.Q. varies hardly at all from year to year, and that the girls who are selected are those who have just failed by a small margin to qualify for a Grammar School place.

Each year some 60 girls of 11 years of age enter the school and are divided into two First Forms alphabetically, and not by any tests of attainment. In 1953, for the first time 32 girls aged 13 were admitted from Secondary Modern schools on the results of another selection examination. This group of girls was known as form IIIR. Meanwhile, the two forms IIIA and IIIB, consisting of 58 girls aged 13 who had been in the school since the age of 11, were divided into two groups, on the basis of mathematical ability, during the first week of the new academic year. The writer, since she was new to the school, took no part in this selection, which was made on the basis of examination and homework marks of the previous year. The mathematics group (referred to as IIIM) consisted of 13 girls from IIIA and 17 girls from IIIB who were taught arithmetic, algebra and geometry. The remaining 28 girls dropped algebra and geometry and were taught arithmetic by another mistress. In theory IIIM would be expected to consist of the brighter members of IIIA and IIIB,

THE SCHEME OF WORK.

Being responsible for the teaching of mathematics throughout the Upper School I had as my youngest pupils the group known as IIIM, and the new IIIR who were to do arithmetic, algebra and geometry. The school worked to a 6-day timetable, and both forms had two lessons each of arithmetic, algebra and geometry during the 6 days. Some of the lessons varied slightly in length according to their position in the timetable, and on paper IIIM had ten minutes more tuition in 6 days than IIIR. IIIM had two mathematics homework periods in 6 days, one of which was always arithmetic while the other was alternatively algebra and geometry. IIIR had three mathematics homework periods in 6 days. At no time was it suggested to IIIR that they should endeavour to catch up with IIIM, and it was never suggested by the headmistress that we should work with this end in view.

The geometry syllabus for Years I to III consisted of the usual 'Stage A' Geometry involving the use of mathematical instruments, construction and properties of angles, triangles, parallelism, triangle and polygon angle sum and concluding with an introduction to congruency and the isosceles triangle theorems. The syllabus for IIIR was shorter but similar, with the omission of congruency and isosceles triangles.

Both forms were taught in exactly the same way i.e., blackboard demonstrations, explanations, and questions and answers from the girls for part of the lesson, and then individual work. Homework was always on the work done in class, and if it appeared from the homework that a point had not been grasped by the majority of pupils then that work was gone over in detail during the next lesson and reworked by the pupils.

During the autumn term IIIM revised their previous year's work, of which they seemed to have forgotten a great deal, and then concentrated on written constructions. The actual construction caused very little trouble but the girls found great difficulty in expressing themselves clearly and concisely so that many tests were given. By the end of this term IIIR had learnt to construct and measure various angles, and to do simple exercises involving these.

By the end of the Spring term IIIR had completed their syllabus, and all except three girls produced satisfactory class and homework. The decision then had to be made, whether to use the Summer term for revision and consolidation or whether to go ahead on the lines of the IIIM syllabus. By this time IIIM had got as far as the simpler theorems. IIIR were settled in the school and had friends in the other two III forms, and were unanimous in their desire to do the same work as IIIM. By the fourth week of the Summer term IIIR, or the majority of them, could do the same exercises as IIIM, but they did not have the same amount of practice at each stage as the latter.

EXAMINATIONS AND RESULTS.

The yearly examinations took place throughout the school in June, and on the 9th June both IIIM and IIIR sat down to the same paper, which was as follows:—

YEAR 3—GEOMETRY.

1. Construct a triangle with the following measurements — AC=6 cms, AB=4·5 cms, BC=7·5 cms. Measure the angles. Write your construction out in full.

2. Prove that if two triangles have two sides of the one equal to two sides of the other each to each, and also the angles included by those sides equal, then the triangles are congruent.

3. Construct an angle of 60°, using ruler and compass only. Bisect it. Describe your construction in detail.

4. Draw a figure to represent three towns, F, G, H, such that G is due west of F, and H bears S. 49°E from G, and S. 25°W from F. Mark the angles on your plan.

5. Draw any triangle ABC and extend BC to D. Prove that angle ACD = angle A + angle B.

RESULTS. (Arithmetic results are given also as a basis for comparison, and their implications are discussed later in this paper).

Geometry was marked out of a total of 50, since it was a combined Algebra and Geometry paper, and arithmetic out of 100. All 32 girls in IIIR were present for both examinations, 29 in IIIM took the geometry examinations, and 28 the arithmetic examinations. On the basis of their term work the absentees would have been expected to obtain about half marks or possibly less.

GEOMETRY

| | RANGE | MEAN | S.D. | N. | AVERAGE AGE (JUNE 9TH). |
|---|---|---|---|---|---|
| IIIM | 5 – 43 | 24·75 | 8·15 | 29 | 14·4 years. |
| IIIR | 7 – 37 | 21·20 | 9·35 | 32 | 14·5 years. |

The difference of 3·55 in the means between the two groups was examined and was found to be not significant, the critical ratio obtained being 1·55. We may, therefore, retain the null hypothesis which asserts that no true difference exists between the two forms.

ARITHMETIC

| | MEAN | S.D. | N. |
|---|---|---|---|
| IIIM | 53·25 | 14·39 | 28 |
| IIIR | 54·95 | 12·68 | 32 |

The mean difference of 1·70 between the two groups was examined in the same way, using Students "t" test, and the result was not significant, the critical ratio obtained being 0·47. Once again we may accept the null hypothesis and say that there is no real difference between the two groups.

DISCUSSION.

It would appear from the statistical treatment of the marks obtained by the two sets of girls that the 32 girls who entered the school at the age of 13 plus have learnt as much geometry or do equally as well in the yearly examinations as the selected group who have had two more years in the school, and consequently two more years' tuition in geometry. It should be borne in mind that the members of IIIR were not grouped according to their ability to do mathematics as they would have been had they entered the school at the age of 11.

These results raise some interesting questions which may perhaps be fruitfully considered by educationalists. On the surface it seems not a little disconcerting that apparently two years' teaching has been unnecessarily expended. But had it? The fact that IIIR have learnt three years' work in one year does not necessarily mean that they will retain it as well as IIIM, who have had more practice at each stage. Unfortunately, I am not in a position to follow this up.

Is the fact that IIIR had geometry homework once every six days, while IIIM had homework every twelve days a contributory factor in this apparent superiority in learning? It may be so, but although IIIM spent less time on homework they had more time devoted to each stage during lessons. It may be asked why I did not push IIIM at the same rate as IIIR, and so make them keep the superiority they started with. The answer is that the majority of IIIM were either not interested in working at speed or were unable to assimilate more than they did. When I began to teach IIIM I found that all but nine girls were bored with geometry and did not wish to go on with it. This attitude persisted well into the Spring term. On the other hand IIIR showed keen interest throughout the year, with the exception of two girls who found the subject completely beyond them and who would obviously have dropped it, had there been any selection at the beginning of the year.

On the basis of past work IIIM could have been expected to gain higher marks in the examination than IIIR, but as has been demonstrated the differences in the two groups of marks is not significant. We cannot assert that IIIR as a group has more intelligence than IIIM. Upon what then depends the fact that after learning geometry for one academic year IIIR caught up with IIIM who had supposedly learnt geometry for three academic years? Had the first two years been wasted, and is there a certain stage of maturation at which a subject needing both abstract and logical thought can most profitably be begun? If this is so, what is the average age at which geometry should be introduced into the curriculum, and do we make a mistake in expecting the 11 year old to tackle it? In these days of ever increasing specialization it is as well to pause and ask such questions, since there may be a minimum age at which the child is intellectually capable of specialising. I have not taught mathematics long enough to have formed an opinion on the best age for beginning geometry proper, but would suggest that for the child who is not as academically inclined as its fellows who enter grammar schools, the age may be as late as 13.

In a recent article Burt (i) points out that according to Binet (with whom he agrees), specific abilities, whilst not separable from general intelligence, are differentiated progressively as the child matures. He produces statistical evidence to support his hypothesis that "between late infancy and early adolescence, there is a definite increase in specialization in ability due, not to education, but to maturation." If this is so, then it would seem that IIIR's performance is due to maturation, i.e., that they were at the stage of intellectual development which enabled them to grasp a new and highly specialised subject and to reach the same standard in one year as their contemporaries who started the subject at the age of 11.

In past years it has been the practice at this school to admit about six girls only at the age of 13, and to put them into the form appropriate to their age, i.e., form III. Members of the staff who had been at the school for several years formed the opinion that on the whole of the late entrants worked harder and were more interested in learning than those who had entered the school at 11, and that by the time they reached Form V they had quite caught up with the others. However, the situation here is probably radically different since these few girls had the incentive of actually working with girls who had been in the school for two years, whereas IIIR did not have this incentive as they were all new to the school. Again in the former case, that of the few late entrants into a well established group, there are two psychologically different groups. It may have been that the girls who entered the school at 11 were disappointed at not getting into a Grammar School, whereas the girls who entered

at 13 could feel that they had been promoted from a Secondary Modern School. Unfortunately it has not been possible for me to investigate this aspect of the subject, and whether the ability of the new entrants at 13 to keep up with the others of their own age is due to maturation or incentive or a combination of the two is a question providing a possible field of research.

Some support for the maturation theory comes from the results of the arithmetic examination. It was not the intention to give both forms the same examination paper, as I did not keep to the order of the prescribed syllabus for IIIR, whereas I did so with IIIM. The syllabus was essentially the same for both forms, but more time was spent in the first term on the Four rules and revision with IIIR in order to find out what they could do since they had come from several different schools. However, when considering examination questions I found that I had covered the same amount of ground with both groups, and therefore gave them both the same paper. It is highly probable that in the year before their entry to IIIR, the girls did more arithmetic in the Secondary Modern Schools than IIIM during the second year in the school under discussion. If this were so we could reasonably expect them to get higher marks than IIIM. The fact that there is no significant difference between the two groups shows that they are not fundamentally brighter than IIIM, in spite of their ability to do as well in the geometry examination although they had studied it for a much shorter period.

Further observations are needed and much more evidence brought before definite conclusions can be made, but it is hoped that the observations recorded here will stimulate curiosity and incite controversy among educationalists, especially those concerned with the teaching of mathematics.

*Reference.*

(i) Sir Cyril Burt — "Differentiation of Intellectual Ability"; Brit. Journal Ed. Psychology: Vol. XXIV, Pt. 11.

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*MAY WE INTRODUCE . . . .*

Our new Chairman, Ian Harris, who has worked with the Association from its early days. A B.Sc. of London University, he is now Senior Mathematics Master of Dartford Grammar School (the school of which he was once a pupil). His main interest is in secondary education, but particularly the dynamic approach to geometry: as a member of the Film Unit he is at present producing a film on *Tangency*.

Our new Secretary, David H. Wheeler, who took an Honours B.Sc. degree in Mathematics at King's College, London. He has taught mathematics in schools at Willesden and Clapham, and is now Senior Mathematics Master of Acton County School.

Our new Editor, Claude Birtwistle, who has also worked with the Association from its foundation, particularly in its work in Northern England. A M.Sc. in Education of London University, he has taught in England and Scotland, and is now Deputy Head of Nelson Secondary Technical School.

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S. O. S.

Mrs. Ogle, the Primary Schools' Representative on the Committee of the Association, is anxious to form a Primary Panel, and those willing to help to develop primary school interests are asked to write to her (address on back cover).

The Association is considering the possibility of compiling a list of applications of mathematics, and the Secretary would like to hear from anyone willing and able to give assistance.

## "WHAT MANCHESTER THINKS TO-DAY . . . ."

Through the courtesy of the Chief Education Officer of Manchester we have been able to examine a report published by that Education Authority on *Mathematics im Manchester Secondary Modern Schools*. The report, intended for head teachers and assistants, is an admirable piece of work which might serve as a model for similar surveys by other authorities. There are many valuable suggestions on the organisation and teaching of the subject, but what is of immediate interest to us is the picture it gives of the state of affairs existing in schools at the moment. Since the survey was conducted over a variety of schools it is a fair assumption that this picture is typical of the country as a whole; as such it should receive our careful attention.

The report makes it clear that the mathematical attainment of pupils entering the schools from the primary schools is not as high as might be expected; many of the more able pupils had not reached their full potential in mathematical attainment. The work in primary schools was mainly arithmetical and concepts of geometry, for example, were relatively unknown. It is disturbing to read that pupils of eleven had no acquaintance with points of the compass, angles and solid figures, while a common complaint appeared to be a lack of knowledge of tables. The general impression given is that the work in the primary schools would benefit from a more practical approach.

On the staffing of the Modern schools the report notes that three-quarters of the teachers engaged in teaching mathematics had no special qualification in the subject, and out of 607 teachers only eight had a degree which included mathematics; under such circumstances a liking for, and skill in, mathematics were not to be easily imparted. However, an even more disturbing feature is the rapid turnover of staff; over the period 1953-56 the percentage of leavers in all subjects was 60, while for those who taught mathematics the figure was 65 per cent. In a subject such as mathematics, continuity is of prime importance, and while the report suggests reasons for this turnover of staff which it regards as 'probably quite unalterable,' until some attempt is made both by Education Authorities and by the teaching profession itself to slow down this movement, there seems little prospect of raising standards of mathematical attainment. Teachers must remember the position they hold is not one merely of having a job; they also have a trust and a moral responsibility for the upbringing of society.

Considering organisation of mathematics, 'cross-setting' was carried out in a number of schools and the time devoted to the subject varied from 2½ to 5 hours a week. Only a few schools had a mathematics room, and in most schools the supply of material for practical work in mathematics was inadequate.

The report suggest that the content of the mathematics course should be widened to include work on measurement, comparison, size, shape, direction and mathematical relationships generally, and points out that many teachers have found that elementary ideas in geometry, algebra and trigonometry are more easily grasped by many children than much of the arithmetic practised to-day. Mathematical experience, in fact, should be the key-note and the report calls for freshness of material and approach. In the first year, for example, revision and extension of primary school work should make way for new and stimulating topics and material.

"What would seem to be the most needed now is for some of the more traditional content of the mathematics syllabuses to be discarded, and for methods to be devised which will encourage modern school children to accept and understand broader mathematical concepts. New approaches which will catch and hold the pupils' interests and more spontaneous practical applications to demonstrate the usefulness of some of the more advanced mathematical ideas, many of which are not themselves inherently difficult, are needed."

It is impossible in a brief review to quote the many excellent suggestions contained in the report, but the ideas are similar to those repeatedly expressed at A.T.A.M. meetings, ideas which should be considered not only by teachers in this type of school, but by all concerned in the teaching of mathematics. These ideas and many others are being developed by writers of a series of articles on 'Secondary Modern School Mathematics' to appear in *Mathematics Teaching*. These articles deserve the careful attention and consideration of all our readers.

C.B.

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## MATHEMATICAL TECHNIQUES IN OPERATIONAL RESEARCH.

S. VAJDA.

Operational Research has been described as a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.

There are now only a few people left—though unfortunately in influential places—who believe that all the mathematics needed for this purpose consists of computations on the back of an envelope. In fact, new disciplines of applied mathematics have been developed to supply the necessary mathematical equipment of Operational Research. These branches would seem to be attractive to newcomers, and examples taken from them might prove stimulating to future operational research workers and teachers alike.

The levels of mathematical sophistication encountered in such problems are diverse, but even simple examples convey the special atmosphere of these new branches. A few of these, though by no means all, will be illustrated in what follows. No knowledge of higher mathematics will be required.

The mathematical techniques of Operational Research are mostly concerned with planning. This becomes necessary whenever there is a limitation on available resources, such as material, money, or labour, and a decision has to be made as to their optimal use.

Let us assume that we have 180 machines of a certain type at our disposal, and that they can perform two different jobs. On the first job, each machine produces per day goods worth £3, on the second job goods worth £2 10s. 0d. It seems clear, from this, that all machines ought to do exclusively the first job.

However, the machines are subject to attrition, and of any number of machines only 1/3 will be available after doing the first job, and 2/3 after doing the second.

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(This paper is based on a talk of the same title, given by the Author at A Short Course For Science and Mathematics Teachers, organised by the Shell Petroleum Co, Ltd., at University College, London, in April, 1958).

On the second day, the same problem is encountered regarding the remaining machines, and again on the third day. We assume that we are not interested in the machines after that. The problem is this: how should the machines be allocated to the two jobs in the successive stages?

Common sense tells us that during the first two days the second job should be done by all machines, and on the third day the first job, since after that we do not care how many machines remain. The following mathematical derivation of this very same result has nevertheless its merits. First, it confirms that there was no flaw in our argument and that we have not overlooked some important point. Secondly, it illustrates the mathematical formulation of such a problem. Thirdly, it shows what techniques are available for its solution even in more complicated cases. These will arise, for instance, if the value of the output or the attrition depend on the number of machines put to either job in a more complex way than linearly, as has been assumed above.

We proceed to formulating our problem algebraically. Denoting the number of machines doing the first job on the first day by $y_1$ the value of the goods produced will be, in £

$$3y_1 + 2{\cdot}5\ (180 - y_1).$$

The number of machines still available is then

$$y_1/3 + 2\ (180 - y_1)\ /3 = A_2, \text{ say.}$$

Of the latter number, we let $y_2$ do the first job and the remainder the other job. The goods produced will be worth

$$3y_2 + 2{\cdot}5\ (A_2 - y_2)$$

and the machines available at the end of the second day number

$$y_2/3 + 2\ (A_2 - y_2)/3 = A_3, \text{ say.}$$

Similarly, introducing $y_3$, the value of goods on the third day is

$$3y_3 + 2{\cdot}5\ (A_3 - y_3).$$

The number of machines remaining now is irrelevant.

The problem is to find $y_1$, $y_2$ and $y_3$ so that

$$3\ (y_1 + y_2 + y_3) + 2{\cdot}5\ (180 + A_2 + A_3 - y_1 - y_2 - y_3)$$

is as large as possible, taking account of the relations between the $y$'s and $A$'s.

To solve the problem, we work backwards, starting with the third day. We have $A_3$ machines available: we do not know how many these are, but we do know that the best we can do with them is to let them all do the first job, i.e.

$$y_3 = A_3 = y_2/3 + 2\ (A_2 - y_2)/3.$$

Now we consider the situation at the start of the second day. The second and third stage together produce

$$3y_2 + 2{\cdot}5\ (A_2 - y_2) + 3y_3 = 4y_2 + 4{\cdot}5\ (A_2 - y_2).$$

Whatever the value of the yet unknown $A_2$, the latter expression will be largest if $y_2 = 0$. This produces

$$4{\cdot}5A_2 = 1{\cdot}5y_1 + 3\ (180 - y_1)$$

for the total of the last two stages.

Now to the very beginning. The total production of all three days is

$$3y_1 + 2{\cdot}5\ (180 - y_1) + 1{\cdot}5y_1 + 3\ (180 - y_1) =$$
$$4{\cdot}5y_1 + 5{\cdot}5\ (180 - y_1).$$

This will be maximised when $y_1 = 0$. The total value of all the goods produced is then £990, and it cannot be made larger by any other choice of the $y$'s. The maximum will be obtained by the following production scheme:

| Day | Machines on Job | | Value in £ |
|---|---|---|---|
| 1 | 180 | 2nd | 450 |
| 2 | 120 | 2nd | 300 |
| 3 | 80 | 1st | 240 |
| | | Total | 990 |

Had we started with any other number than 180, the procedure would be the same, but if fractional values had been encountered, some common sense adjustment would have to be made.

The procedure outlined above is typical for the techniques of Dynamic Programming (1). But because we have chosen constant output and attrition rates, we can solve this particular problem also by Linear Programming.

After substituting for the $A$'s, the expression to be maximised can be written as follows:

$$950 - 8y_1/9 - \tfrac{1}{3}y_2 + \tfrac{1}{2}y_3 = B, \text{ say}$$

Because the y's must be non-negative, and cannot exceed the corresponding $A$'s, they are subject to the following constraints:

$$0 \leq y_1 \leq 180$$
$$0 \leq y_2 \leq 120 - \tfrac{1}{3}y_1$$
$$0 \leq y_3 \leq 80 - 2y_1/9 - \tfrac{1}{3}y_2.$$

All these formulae are linear, and hence the name Linear Programming for such problems. We cannot exhibit here exhaustively the technique for solving them (2), but in the present case the answer can be found by inspection. B will be largest if $y_1$ and $y_2$ are as small, and $y_3$ as large as possible, because in the expression for B, the coefficients of the first two variables are negative, and that of the last positive. Now the first two cannot be smaller than 0, and the last not larger than 80. These values happen to be compatible (hence the solution by inspection is possible), and this is the final answer, as we already know.

Many other problems can be tackled by Linear Programming and Ref. (3) contains examples of its applications.

Game Theory is another important technique for the worker in Operational Research, though more valuable for its mental attitude rather than for practical results. *The New Gamesmanship* was described in an article by T. J. Fletcher in this periodical, and we shall here merely exhibit the connection between Game Theory and Linear Programming. As an illustration, consider the pay-off table

| | | |
|---|---|---|
| 1 | − 1 | 3 |
| 3 | 5 | − 3 |
| 6 | 2 | − 2 |

Player A wishes to find non-negative values $x_1$, $x_2$ and $x_3$ such that $x_1 + x_2 + x_3 = 1$ and that the minimum, say $v_1$, of

$$x_1 + 3x_2 + 6x_3, -x_1 + 5x_2 + 2x_3, \text{ and } 3x_1 - 3x_2 - 2x_3$$

is as large as possible. This means that

$v_1$ should be as large as possible, subject to

$$x_1 + 3x_2 + 6x_3 \geq v_1$$
$$-x_1 + 5x_2 + 2x_3 \geq v_1$$
$$3x_1 - 3x_2 - 2x_3 \geq v_1$$
$$x_1 + x_2 + x_3 = 1.$$

This is a linear programming problem.
Player B, on the other hand, wishes to find non-negative values $y_1$, $y_2$ and $y_3$ such that $y_1 + y_2 + y_3 = 1$ and that the maximum, say $v_2$, of

$$y_1 - y_2 + 3y_3,\ 3y_1 + 5y_2 - 3y_3,\ 6y_1 + 2y_2 - 2y_3$$

is as small as possible. This means that

$v_2$ should be as small as possible, subject to

$$y_1 - y_2 + 3y_3 \leq v_2$$
$$3y_1 + 5y_2 - 3y_3 \leq v_2$$
$$6y_1 + 2y_2 - 2y_3 \leq v_2.$$

These two linear Programming problems are "dual" to each other, and the celebrated Main (Minimax) Theorem of the Theory of Games proves that max. $v_1 =$ min. $v_2$.

The solution of the present problem is

$$(x_1, x_2, x_3) = (\tfrac{2}{3}, \tfrac{1}{3}, 0),\ (y_1, y_2, y_3) = (0, \tfrac{1}{2}, \tfrac{1}{2})$$

and max $v_1 =$ min $v_2 = 1$, the so-called "value" of this game.

The Minimax Theorem can be illustrated geometrically, provided neither of the players has more than three pure strategies from which to choose (see figures).

To illustrate the present example with three pure strategies of either player, imagine an equilateral triangle and construct, above it, a plane through points of heights 1, 3, 6 above the vertices of this triangle. If B chooses his first strategy, and A chooses either his first, or his second, or his third, then these heights indicate the respective payments from B to A. Also, if A chooses strategies in any proportions, and determines the point with these barycentric co-ordinates in the basic triangle, then the height of the plane indicates again the pay-off to him, provided B has chosen his first strategy. For B's second and third strategies similar planes can be constructed. It is A's aim to find that point in his basic triangle which makes the height on the lowest plane above it as large as possible. In other words, he wants to find the highest point on the floor of the model. Similarly, B can construct such a model and will want to find the lowest point on the ceiling.

The Minimax Theorem now reduces to this:

There are two models of three planes above a basic triangle. Imagine vertical rods from all six vertices. On each rod in one of the models there will be three intersections with planes, and their heights are the same as those of the intersections of one plane with the three rods in the other model. The lowest point of the ceiling of the one has the same height as the highest point of the floor of the other.

It would be interesting to find a purely geometrical proof of this theorem.

It is hoped that the description of these selected operational research topics shows that the new branches of applied mathematics have some interest, and that it encourages young people, interested in mathematical activities, to choose jobs where they can satisfy this interest and even contribute to the development of the subjects. Operational research departments in industry as well as in the civil service have much to offer which is of interest and of promise to mentally active young mathematicians.

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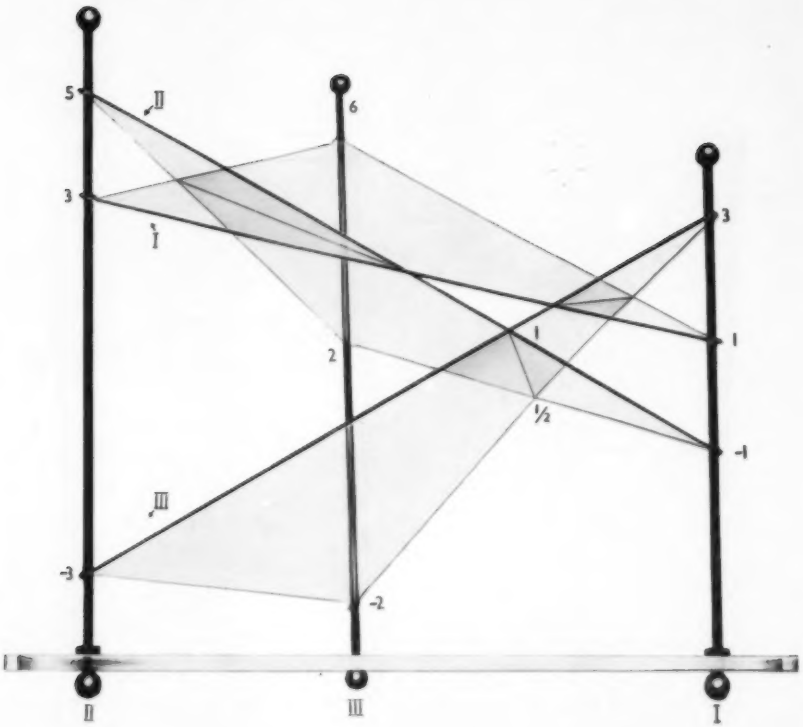
(1) Richard Bellman, Dynamic Programming, Princeton University Press, 1957.
(2) S. Vajda, The Theory of Games and Linear Programming, Methuen, 1956.
(3) S. Vajda, Readings in Linear Programming, Pitman, 1958.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Mr. J. W. PESKETT, B.Sc., The Royal Military Academy, Sandhurst

Jack Peskett was at the inaugural committee meeting of the Association in 1952, and since then he served it with a devotion and zeal which it would be hard to equal. We remember early meetings at Braziers Park where he conducted seminars on model making, and demonstrated his wide range of apparatus with that quiet, quizzical good humour that was so characteristic. He became our expert on models, and during the next seven years there was scarcely a demonstration of ours, anywhere in England, that did not profit from his labours. The exhibitions of models and apparatus were nearly all mounted by him, and on the few occasions when he was not present in person, he was always represented by his work.

His reputation extended beyond this country, as his models were admired in Madrid in 1957 and he contributed a chapter to the International Commission's book on teaching apparatus. Like all modellers he made his models primarily for the joy of doing so, and only secondly for the instruction which they gave others. But in this way he taught better than he knew, for he bore witness to the satisfaction which craftsmanship can bring, and inspired others to seek it in their own mathematical work. Above all he sought models that moved and that embodied an idea. Practical matters were more congenial to him than philosophical discussion, and he expressed himself in actions when others often fumbled for words.

The room in which he taught at Sandhurst was surrounded by pieces of equipment for teaching mechanics, and in the Association we lost much because, keeping him so busy with exhibitions, we never saw him give a lesson.

A short while ago we learnt that Jack Peskett was ill, and we did not realise at first how serious it was; but he died of cancer in hospital at Guildford, on 14th March, after a final illness that was, mercifully, not unduly long. We extend our deepest sympathy and condolences to his wife and son.

Dying at an age when men usually expect many years of happy, active life, he leaves an irreplaceable gap in our ranks. We had for him the deepest affection and regard, and he was to have been our next President—the greatest honour we had to give. Our lives are richer for knowing him, and we must labour to carry on the work which he showed us how to do.

As a memorial to Jack Peskett the Association has opened a fund to make a donation to the Imperial Cancer Research Fund. Members and friends may send contributions to Miss B. Briggs, 8 Glencairn Crescent, Edinburgh, 12.

## SIXTH ANNUAL GENERAL MEETING.

At the meeting on February 14th, 1959, the Secretary (Mrs. R. M. Fyfe) reported a paid up-membership of 1058 for the year 1958, an increase of about 200 during the year. The year was notable for a very full programme of Day Conferences and preliminary work for the formation of branches.

The Treasurer announced a deficit on the year's working of £88 5s. 6d. and introduced a budget which would reduce this to about £30 by the end of the year.

The Constitution of the Association was amended to put into effect the raising of the annual subscription to ten shillings and to make changes in the numbers of officers and committee members.

The resignation of Mrs. Fyfe as Secretary was received with regret and with warm thanks for the work that she had performed during her two years of office.

The altered version of the Constitution is given below, and the list of Officers and Committee members appears on the cover of this issue.

D.H.W.

## ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS

### CONSTITUTION

1. The name of the Association shall be "The Association for Teaching Aids in Mathematics."

2. The aims of the Association shall be to promote the study of the teaching of mathematics, and to improve the aids used therein, through week-end courses, exhibitions, demonstration lessons, and the publication of a bulletin.

3. Membership of the Association is open to all who are interested in these aims.

4. The subscription shall be ten shillings per annum, covering the calendar year, and shall be payable to the Treasurer from the beginning of the year. Subscriptions of new members joining after October 1st shall also cover membership for the following calendar year.

5. The Officers of the Association shall be a President, a Chairman, two Deputy Chairmen, an Honorary Secretary, an Honorary Treasurer and an Executive Editor. The Committee shall consist of these Officers and six other elected members together with representatives of local Branches, Panels and Production Units. The Committee shall have power to co-opt four additional members and to delegate responsibility for particular aspects of the work of the Association. The Committee shall draw up standing orders and rules of procedure.

6. The Committee of the Association shall call an Annual General Meeting once in the course of each calendar year. The Annual General Meeting shall have power to fill the Offices of President, Honorary Secretary, Honorary Treasurer, to elect the ten other members of the Committee and to approve the appointment of Vice-Presidents. The Committee shall have the power to fill any vacancy which may occur during the current year. Nominations, duly proposed and seconded, and with the consent of the nominee, must reach the Secretary, in writing, at least 28 days before the date of the Annual General Meeting.

7. Extraordinary General Meetings may be called at any time, either by the Committee, or at the written request of a least twenty members of the Association, with reasonable time allowed for making the necessary arrangements.

8. A quorum for a General Meeting shall be twenty members.

9. The Constitution may be amended by a resolution passed by a simple majority at an Annual General Meeting. Notice of any such resolution must reach the Secretary in writing at least 28 days before the date of the Meeting.

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## ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS
## SUMMARY OF ACCOUNTS FOR YEAR ENDING 31st DECEMBER, 1958.

### EXPENDITURE

| To | £ | s. | d. |
|---|---|---|---|
| Printing Bulletins | 400 | 1 | 7 |
| Postage on Bulletins | 53 | 14 | 6 |
| Secretary: stationery, postage, etc. | 51 | 3 | 8 |
| Treasurer: stationery, postage | 3 | 0 | 0 |
| Editor: stationery, postage | 5 | 19 | 4 |
| Committee Travelling | 34 | 0 | 10 |
| Printing | 7 | 18 | 6 |
| Film Purchase | 6 | 0 | 0 |
| Meetings Expenses | 35 | 7 | 0 |
| Miscellaneous Items | 1 | 13 | 8 |
| | 598 | 19 | 1 |

### INCOME

| By | £ | s. | d. |
|---|---|---|---|
| Subscriptions | 306 | 1 | 0 |
| Sale of Bulletins | 12 | 9 | 0 |
| Advertisements in Bulletins | 12 | 0 | 0 |
| Miscellaneous Sales | 13 | 4 | 0 |
| Meetings Receipts | 42 | 6 | 0 |
| Donation: Mathematical Pie Ltd. | 100 | 0 | 0 |
| Donation: Institute of Education Cambridge | 10 | 10 | 0 |
| Other Donations | | 10 | 0 |
| Deficit, being excess of Expenditure over Income | 101 | 19 | 1 |
| | 598 | 19 | 1 |

### BALANCE SHEET

| LIABILITIES — To | £ | s. | d. | ASSETS — By | £ | s. | d. |
|---|---|---|---|---|---|---|---|
| Balance, Dec. 31st, 1957 | 112 | 12 | 11 | Cash at Bank | 10 | 13 | 10 |
| Deficit, Dec. 31st, 1958 | 101 | 19 | 1 | | | | |
| | 10 | 13 | 10 | | 10 | 13 | 10 |

OUTSTANDING DEBT, Dec. 31st, 1958
Cost of Bulletin, No. 8 .. .. .. £98 19 4

DEFICIT .. .. .. .. .. £88 5 6

B. I. Briggs, *Hon. Treasurer.*

Above is a summary of the general accounts, audited and found correct by Ian Harris, 5.1.59.

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# PRODUCTS, POWERS AND ROOTS IN CUISENAIRE NOTATION

A. G. SILLITTO

(Note: This article is written on the assumption that the reader will actually perform the suggested manipulations with the rods).

The characteristic Cuisenaire notation for the operation of multiplication of one number by another is a "cross" consisting of the rod denoting the multiplier placed across the rod denoting the multiplicand. Thus with the white rod as unit, $8 \times 6$ (which I take here to mean 8 multiplied by 6) will be symbolised by the brown rod "crossed" by the dark green rod—a "brown $\times$ dark-green cross." The notation is suggested by the fact that six eights is represented, in the addition sense, by a train of six brown rods, and these, if placed side-by-side, could just be spanned by the dark-green rod. Also the equivalence of two products, e.g. crimson $\times$ light-green and dark-green $\times$ red, is indicated by the fact that the three crimson rods side-by-side can be rearranged to cover exactly the two dark-green rods—for example by laying both sets out in "trains."

But what if we decide to take the *red* rod as unit? Then the brown $\times$ dark-green cross must symbolise the multiplication $4 \times 3$, with product 12 (and no longer the multiplication $8 \times 6 = 48$). Yet this product-symbol must still mean "as many brown rods as, side-by-side, can be spanned by the dark-green rod." These, laid end-to-end, would be equivalent to 48 whites: but we recall that *red* is the unit; we would require 24 of these; have we then "proved" that $4 \times 3 = 24$?

To clarify our ideas, let us form the cross "red $\times$ red." As red is the unit, this cross symbolises $1 \times 1$; and *this product is certainly* 1. Our system of measurement is therefore the following: Rods arranged linearly represent numbers in terms of red as unit; but rod-products have to be evaluated in terms of a unit-product symbolised by the red $\times$ red cross, whose magnitude is that of two red rods side-by-side (or end-to-end).

And so our brown $\times$ dark-green *product* falls to be evaluated in terms of the red $\times$ red product; and we can confirm, either by a linear rearrangement, or by arranging *pairs* of reds to cover the browns and noting that we need *twelve* such pairs, that the product of $4 \times 3$ is indeed 12.

This interpretation of the notation is perfectly trustworthy. If it suits us to say: "If light-green is the unit, dark-green is 2; and if red is the unit, brown is 4," then we may accept the implication that the brown $\times$ dark-green cross now symbolises $4 \times 2$, and is to be evaluated in terms of the unit-product which is the red $\times$ light-green cross. Now the brown rods which, side-by-side, can be spanned by the dark-green can in fact be exactly covered by the 8-fold repetition of the set of red rods which, side-by-side, can be spanned by the light-green.

This last example, with its suspect-looking use of two different units of linear measure, may be interpreted in various quite sensible ways. For example, if the red measures 1 *volt*, brown must measure 4 volts; and if light-green measures 1 *amp*, dark-green measures 2 amps. Then the unit-product cross, red $\times$ light-green, measures 1 *watt*; and the brown $\times$ dark-green product cross measures the wattage at 4 volts and 2 amps. (It is of course open to the objection that we may be tempted, by comparing our unit rods, to assert that 1 amp $= 1\frac{1}{2}$ volts. But we resist this temptation, so the comparison just is not made).

*Multiplication of fractions.*

This idea can be applied to the mutiplication of fractions. (In the Cuisenaire literature, there is some discussion of the operation of finding a fraction of a fraction. But what is proposed here is that we simply rely on our interpretation of the cross notation for multiplication, as defined for integers, to provide consistent results for products of fractions).

*Example* 1. $\frac{2}{3} \times \frac{2}{3}$

Take light-green as unit. Then red is $\frac{2}{3}$, and the red $\times$ red cross symbolises $\frac{2}{3} \times \frac{2}{3}$. The unit-product is symbolised by the light-green $\times$ light-green cross. Thus the value of $\frac{2}{3} \times \frac{2}{3}$ is the measure of the red $\times$ red cross, in terms of the light-green $\times$ light-green cross as unit. Referring back to the original interpretation of crosses, red $\times$ red = crimson, and light-green $\times$ light-green = blue; so $\frac{2}{3} \times \frac{2}{3}$ is "crimson measured in terms of blue," or 4/9.

*Example* 2. $2/5 \times 3/5$

Using yellow as unit, we have to consider red $\times$ light-green in terms of the unit-product yellow $\times$ yellow. Referring back to our knowledge of the products of integers (*or* going back even further back and noting that the 5 yellow rods side-by-side can be covered by 25 whites, and the three reds side-by-side by 6 whites), we obtain the value of the product, 6/25.

*Example* 3. $2/5 \times 4/7$

Here we use two different units. (If this is objected to, it is perfectly possible to write $2/5 \times 4/7 = 14/35 \times 20/35$, and use the techniques learned in examples 1 and 2).

If yellow is unit, red is 2/5. If black is unit, crimson is 4/7. Then $2/5 \times 4/7$ is symbolised by the red $\times$ crimson cross, in terms of a unit-product which is the yellow $\times$ black cross. Thus by argument as in example 2, the product is 8/35.

*Division of a fraction by a fraction.*

*Example* 4. $8/35 \div 4/7$

In example 3 we found that the product of the fraction denoted by red: yellow, by the fraction denoted by crimson: black, was the fraction denoted by red $\times$ crimson: yellow $\times$ black. In fact, to multiply the fraction red: yellow by crimson: black, we supplied a factor crimson to the numerator-cross and a factor black to the denominator-cross. The process of division is simply the inverse of this. To divide the fraction symbolised by red $\times$ crimson: yellow $\times$ black by the fraction symbolised by crimson: black, *remove* a factor crimson from the numerator-cross and a factor black from the denominator-cross.

*Example* 5. $8/9 \div 5/7$

Denote 8/9 by rods brown: blue. To perform the "removals" suggested in example 4 we must first ensure that the factors are present. Equivalent to the fraction brown: blue is the fraction brown $\times$ yellow: blue $\times$ yellow, and, further, the fraction brown $\times$ yellow $\times$ black: blue $\times$ yellow $\times$ black. To divide this by 5/7, i.e. by yellow: black, we remove a factor yellow from the numerator-cross and a factor black from the denominator-cross. This leaves brown $\times$ black: blue $\times$ yellow; i.e. $8/9 \div 5/7 = (8 \times 7)/(9 \times 5)$.

*Powers and Roots*

In an obvious sense, the triple-cross, or "tower," yellow $\times$ yellow $\times$ yellow, may be taken to signify "yellow-cubed"; and the product of yellow-squared and

yellow-cubed will be a 5-tuple yellow cross; and the square of yellow-cubed will be a 6-tuple yellow cross.

If now we take the triple-yellow cross to represent 5, then the yellow rod alone must represent the cube-root of 5, and the yellow × yellow cross must represent the square of the cube-root of 5.

Also $5^2$ will be a 6-tuple cross of yellows; and we note that this contains the cube of the yellow × yellow cross. Hence the yellow × yellow cross represents also the cube-root of the square of 5.

To evaluate the product of the cube-root of 5 by the square root of 5, we represent 5 itself by the 6-tuple yellow cross: so the cube-root of 5 is now the yellow × yellow cross and the square-root of 5 is the triple-yellow cross; and the product is the 5-tuple yellow cross which (we know) signifies either the fifth power of the sixth-root, or the sixth-root of the fifth power, of 5, by the argument of the two preceding paragraphs.

To evaluate the product of the cube-root of 5 by the cube-root of 4, take the triple yellow-cross to represent 5, and the triple crimson cross to represent 4. Then the cube-roots are represented by the single yellow and single crimson rods, and their product as usual by the yellow × crimson cross. To interpret this we note that the 6-tuple cross "triple-yellow by triple-crimson" signifies 5 × 4: and a rearrangement of its factors reveals this to be the cube of the yellow × crimson cross which therefore represents the cube-root of 5 × 4. Conclusion: cube-root of 5 × cube-root of 4 = cube-root of 20.

*An index notation*

Just as the product of "as many yellows as, side-by-side, can be spanned by a light-green" can be signified by a conventional—and preferably suggestive—arrangement of a yellow and a light-green rod—in fact, the usual "cross"—so the multiple cross consisting of as many yellow factors, one above the other, as can be spanned by a light-green rod standing on end, can if we choose be signified by a light-green rod standing upright beside a yellow rod. If we take white as unit, then this symbol stands for 5 to the power 3.

If we now take the triple-yellow cross to represent 5, its sign must still be upright-light-green-besides-horizontal-yellow; and the meaning of this is 5 to the power 1; vertically, light-green is the unit. With the same argument, the yellow × yellow cross—which we agreed to mean both the square of the cube-root of 5 and the cube-root of the square of 5—will be signified by "upright-red-beside-horizontal-yellow." Now as the red is two-thirds of the light-green which is the unit vertically, it is quite consistent to regard this as 5 to the power $\frac{2}{3}$. Also we see at once that since (for example) brown is $\frac{2}{3}$ of orange + red, we can if we choose change the symbol for 5 to the twelve-fold yellow cross, or "upright-orange + red beside-horizontal-yellow," and the square of the cube-root of 5 will be symbolised by "upright-brown-beside-horizontal-yellow." Such equivalences will of course allow of such multiplications as 5 to the power $\frac{2}{3}$ by 5 to the power $\frac{1}{4}$; the multiplication being effectively performed by the end-on addition of vertical brown and light-green rods. (Note that it is instructive to compare this procedure with the discussion, above, of the multiplication of the cube-root of 5 by the square-root of 5).

The same type of argument, which it is unnecessary to detail, applies to such a problem as "Raise 5 to the power $\frac{2}{3}$, to the power $\frac{3}{5}$."

## LIVERPOOL MATHEMATICS CONFERENCE, Easter, 1959.

Of the 150 who attended, two-thirds were in residence at Derby Hall, and the others joined the morning sessions held in the new hall of the medical school of the University. The ratio of teachers to representatives in industry was about 4:1, the ratio of men to women 5:1. Many applicants had had to be refused. The conference was a successor to the Oxford Conference of 1957; another may follow in 1961, and Prof. Barnard of Imperial College undertook to convene a committee.

The charge for the course was nominal, Industrialists having been persuaded by those in managerial posts who take a broad and realistic view of the critical position of Britain's future that it was worthwhile to make financial aid available; it is greatly to their credit that thereby such an important Conference resulted, but participants were aware also of the valuable contribution made by Liverpool University. The weakness of democracy, as everyone knows, is that it is not so easy (as, for example in U.S.S.R.) to mobilise the forces available and deploy them towards an end in view: making detailed "Plans" of progress without having recourse to compulsory measures.

The lectures were drawn in about equal numbers from the teaching side, and from those not working in schools and universities. A comprehensive choice of subject matter gave information about the many ways in which mathematics is applied in scientific fields of research, and its use in the everyday life of organisation and production, in large-scale industries.

The keynote throughout, and made explicit by Dr. R. Beeching in his address on the last morning, was the necessity of making "mathematical models" through which a problem could be solved. It was pointed out, time and time again, that the danger was not so much of then getting a wrong solution, but of finding that the wrong problem had been posed. In other words, as is well known, that what matters is to ask the right question.

Prof. L. Rosenhead, the Conference Chairman, summed up the aims of the Conference by saying that children must be sheltered and protected from the difficulties that they will have to face as adults but that there is no reason why teachers should be protected. The staffs of universities, he thought, did live a very sheltered life, and, to a slightly lesser extent, the same was true for the teachers in the schools. This is a fair criticism for we do, as a body, tend to become so absorbed and identified with our day to day difficulties and humdrum tasks that there is no energy available for surveying the whole problem.

What the situation demands is that *twice* as many young people should be found who are competent to do jobs which require the use of mathematics at different levels. This amounts to a revolution, and it seems reasonable to seek a revolutionary solution. It was clear from the discussions that the minor improvements (of status, salary, etc., and ways of finding new recruits to the profession) envisaged by teachers, only brush the fringe of the problem and cannot effect the hoped for large-scale changes needed to meet the urgent demands of a technological world and acute international competition from, as Dr. Hammersley said: "Other countries which also have man-power and brains."

If the awareness of the teachers present has been raised by the wide view so ably presented at Liverpool, the next step should be that they get together at all levels to see what can be devised. Their sphere of power, and their clear responsibility, is to find out how teaching can be improved and how more effective learning can take place. It was hinted that a lead might come from the Mathematical Association, or from the Ministry, or from Universities; also that Industrialists might be persuaded to give aid and encouragement if they were convinced of the sincerity and value of proposed efforts to meet the problem.

Our own Association was formed by those who believe that it is the teachers who do exist and who daily meet the children and students in the classroom, who must improve themselves, believe in their competence to do so and in the creative potential of all children.

It was a privilege and a stimulus to attend such a Conference, and to come in contact with the intellectual prowess, and wise judgment, of the "top flight." We can repay the debt by seizing the opportunity *at once* to make sure that many more of our pupils, in all kinds of schools and from the ages of 5 – 19 years of age, find that the mathematics we present to them is of absorbing interest.

R.M.F.

(The conference proceedings are to be published by the Times Publishing Company, price 7s. 6d.)


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# MAKE YOUR OWN FILM-STRIPS

CLAUDE BIRTWISTLE

In the issue of *Mathematics Teaching* for April, 1958 (No. 6) an article by Mr. C. Hope described the way in which film-strips could stimulate thought and activity in the mathematics lesson. Those who have used film-strips will need no convincing of their effectiveness in classroom teaching; those who have not are ignoring a valuable aid to their work. But all would do well to read again Mr. Hope's article and study the many novel ways of using a film-strip which he suggests. It may be, however, that some will find difficulty in obtaining film-strips containing exactly what they want in order to teach a particular topic. The film-strip, in fact, like other teaching aids, must be a rather personal thing and be very largly adapted to one's own teaching style; surely no two teachers use any one film-strip in exactly the same way. Consequently it would appear that the ideal is to make one's own film-strips. Various firms undertake the production of strips from diagrams and photographs submitted to them, but the production of one's own film-strip is not a very difficult procedure, especially for those with an interest (however slight) in photography.

The main essential is a 35 mm. still camera. This type of camera has become the most popular size during recent years because of the advent of colour photography on 35 mm. film; after exposure and processing the film is returned as colour slides which are projected or viewed through hand-viewers. While the more expensive cameras of this type may cost hundreds of pounds, the development of colour photography has led manufacturers to produce cheaper versions. Some very excellent ones are to be found for around twenty pounds, while quite effective models may be purchased for as little as ten pounds. With any of these cameras it is possible to produce one's own film-strips.

The first essential is to purchase a supplementary close-up lens. The cost is about five shillings unmounted (the lens will fit an existing filter-holder); the cost mounted is about fourteen shillings. These supplementary lenses are available to cover different ranges and there is one which enables the camera to be focused on objects about ten to twelve inches away from it. A leaflet accompanying the lens gives a conversion table for focusing by use of the existing camera focusing-scale. Focused down to ten inches the camera covers a drawing or photograph approximately nine inches by six inches. In practice the drawings are placed on a table with the camera held rigidly above the table pointing down on to it. Rigidity is important as exposure is not short and any vibration naturally spoils the photograph. Keen photographers with an enlarger may use the enlarger column and arm to hold the camera.

The view finder does not operate at these close distances, and it is essential to position the sheets to be photographed so that their centre lies directly below the camera lenses.

It is essential to use a fine-grain film such as Ilford Pan F or F.P.3. This is loaded into the camera in the usual way, a few blank exposures are made to provide the 'leader' for feeding through the projector, then the camera is mounted above the drawings or photographs which are exposed in sequence. The exposure depends upon circumstances; ideally artificial lighting should be used, thereby ensuring uniformity, but even here a little initial experiment is necessary. With

two Photofloods exposure is about $\frac{1}{8}$ sec., at f/11 on Pan F film. However, artificial light is not necessary and exposure can be made in daylight near any bright window. It is more difficult to suggest correct exposure in these circumstances as the brightness of the room, time of day, cloudiness, etc. affect the calculation. As a start to experiment, however, an exposure of $\frac{1}{4}$ sec. at f/8 on F.P.3. should yield some results. A few test exposures should be made and records kept so that the correct exposure may be ascertained. With exposures such as these it is desirable to use a cable release for the shutter to prevent camera shake.

Lines and lettering on diagrams should be bold; on a drawing of size nine by six inches, lines of about $\frac{1}{16}$ inch thickness give good reproduction. Usually the best medium to use is black Indian Ink on white paper. As the projected diagram gives best results on the screen when it appears as white lines on a black ground, the use of black ink on white paper enables one to develop the film in the normal way and use this negative film as the film-strip. This means, of course, that there is only one copy, but as the original drawings are available it is no more trouble to expose and develop further films as required than it would be to make extra strips by printing from a negative strip. In processing the strip a fine-grain developer should be used (Kodak formula D 23 is simple and effective), and if the strip is being processed professionally this should be stated clearly when handing in the film. (It should also be emphasised that prints are not required and the film is not to be cut).

While this method is easy for diagrams, the reproduction of pictures and other objects is not possible simply by using a negative film; such pictures on the strip would have blacks replaced by whites and vice versa. To overcome this difficulty there is a method of processing by which the pictures on the original film are made into positive pictures, i.e. black and white as in the original photograph or drawing. The procedure, which is known as reversal processing, needs a little more time and trouble than straight-forward development, but can be easily undertaken by anyone used to developing one's own films. The method is given in full in the Ilford Manual of Photography or may be obtained in leaflet form through photographic dealers as Ilford Technical Information Sheet T 203. In outline the process consists of development followed by washing and immersion of the film in a bleaching bath; this bleaches out the initial (negative) image. After washing and soaking in a clearing solution, the film is given a second exposure under a 100 watt electric lamp. This produces a reversed (i.e. positive) image, which is then developed and fixed. The whole process takes about half an hour. Results are usually very good provided care is taken to give absolutely correct exposure of the film in the camera, and to control the time and temperature of the first development.

For those who do not possess the facilities to process their own films but wish to make film-strips by reversal process, Gevaert market a 35 mm. black and white film which is returned to them for processing (the processing charge is included in the initial cost of the film) and is returned to the customer reversal processed as a positive film-strip. Supplies are obtainable from most large photographic dealers.

Film-strips made by the methods described above are of double-frame size, i.e. 24 by 36 mm., and it is possible to include about thirty frames on one film-strip.

The films are normally for thirty-six exposures, but a length of blank film should be left at each end to fasten in the projector. The cost of producing strips by these methods is about six shillings a strip.

Once one has produced black and white film-strips satisfactorily, there is no reason why attempts should not be made with strips in colour. The method of exposure is exactly the same as in the case of monochrome films, but the cost of film is much greater. Most colour films are, of course, trade processed and it is necessary here to make it quite clear when sending the film that it is to be returned as a strip and must not be cut.

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(*Mathematics Teaching* is always pleased to answer any technical enquiries arising out of articles such as the above, or to assist in resolving difficulties which readers may meet in constructing models and other teaching aids. Enquiries should be addressed to the Editor and be accompanied by a stamped addressed envelope).

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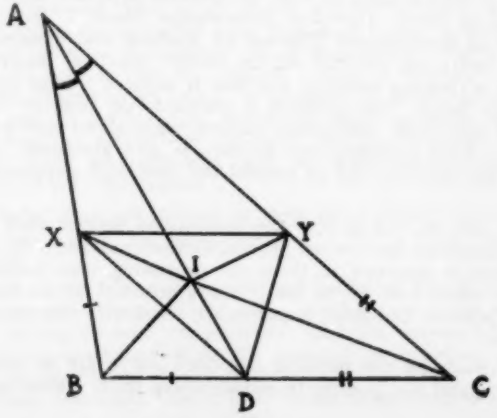
## ANGLE BISECTORS

DAVID H. WHEELER

Many of the readers of this bulletin will have seen Nicolet's film of the internal angle bisector of a triangle. (It was used in a demonstration lesson by Mr. R. H. Fielding at the meeting of the Middlesex Branch of the Association on October 4th). This film demonstrates dynamically that when the segments into which an internal angle bisector divides the opposite side are rotated to fall along the other two sides of the triangle, the join of their extremities is parallel to the divided side. A proof of this property is trivial if the usual theorem about division of sides by angle bisectors is assumed. It is of interest, however, to see that this proposition can be proved without using statements involving ratios, thus making it possible to derive the bisector theorem from this situation.

It is given that AD bisects $\angle BAC$ and that $BX = BD$, $CY = CD$.

We show that XY is parallel to BC.



Let BI, CI bisect $\angle ABC$, $\angle BCA$. Join XI, XD, YI, YD.
Then BI, CI bisect XD, YD at right angles.
$\therefore$ I is the circumcentre of $\Delta XYD$.

$$\therefore \angle XIY = 2 \angle XDY$$
$$= 2\,[180^\circ - (90^\circ - B/2) - (90^\circ - C/2)]$$
$$= B + C$$

$\therefore$ A, Y, I, X are concyclic points.

$$\therefore \angle AIY = \angle AXY$$

But, from the equiangular triangles AIY, ABD.

$$\angle AIY = \angle ABD$$
$$\therefore \angle AXY = \angle ABD.$$

These being corresponding angles.
XY is parallel to BC.

This proof is not the simplest possible, but utilises a configuration with many points of interest in which the discovery of the parallelism of XY and BC is almost incidental. If the circle YXD is drawn, congruent isosceles triangles are formed by joining I to the pairs of intersections of the circle with the sides of triangle ABC. If the circle intersects AB at X, P and AC at Q, Y, then PQ and XY intersect at R on AI and the quadrilaterals IPXR, IYQR are congruent and cyclic. These facts can also be used to prove that XY is parallel to BC.

A more elementary proof which does not depend on a knowledge of the concurrence of the angle bisectors or of properties of cyclic quadrilaterals (and is far less interesting, though probably better for the job immediately in hand) is also given here.

Let $p_1$, $p_2$ be the perpendiculars from D to AB, AC and $q_1$, $q_2$ the perpendiculars from X, Y to BC.
Then, by congruency, or from the area of $\Delta XBD$,

$$p_1 = q_1$$

Similarly, $p_2 = q_2$
But $p_1 = p_2$ (locus property)
$\therefore$ $q_1 = q_2$
Whence XY is parallel to BC.

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## PROGRAMMING LOGARITHMS

Technical examinations frequently ask for given values to be substituted into a formula which leads to computing by logarithms fractions like (i) $\dfrac{(3{\cdot}14)^2}{\sqrt{7{\cdot}39}}$

The usual oral work produces 'from twice the log of 3 I subtract half the log of 7, and antilog the result.' The tedium of practising these computations may be diminished slightly for children by asking them to invent a notation for operations.

Example (i) is described 2 L 3 <— – ½ L 7.
<— – x means 'from the answer so far, subtract x', and therefore —> – x means 'subtract the answer so far from x.' This arrow is a useful alternative to complex brackets in longer examples. The full stop is a convenient sign for 'antilog.'

Examples like (ii) $\sqrt{\dfrac{(5{\cdot}3) + \sqrt{\cdot 51}}{\sqrt[3]{2 \cdot 1}}}$ require a storing symbol S, and a pause symbol P, denoting the start of a new sequence. The programme for (ii) is

$2\ L\ 5\ .\ S_1\ P\ \tfrac{1}{2}\ L\ {\cdot}5.\ S_2\ P\ S_1\ +\ S_2\ S_3\ P\ \tfrac{1}{3}\ L\ 2\ S_4\ P\ L\ S_3\ <\!\!-\ -\ S_4\ <\!\!-\ \tfrac{1}{2}.$

Other variations of this notation can be easily invented by children. The best are those which follow the usual column layout, and are unambiguous. Perhaps the most valuable feature of this exercise is the detection and avoidance of ambiguity. It could lead to discussion on programming for electronic digital computers, and the translation of operating symbols into mark/space combinations.

P. G. HARRISON.

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## PROBLEMS AND LEARNING

R. M. FYFE

Will those who intend to read this article please begin by solving this problem before reading further:

A father and son have both just had birthdays. Four years ago, the father's age was the square of what his son's age is now. Now however, the father is twice as old as his son will be in fourteen years' time. How old are they now?

The question to be discussed is *how* problems are solved.

Will the reader next ask himself whether the method he used was intelligent or automatic. Then, how much knowledge of everyday life is relevant, and what mathematical facts could give clues. Is it an advantage or a disadvantage to find a method of some generality, e.g. equations, that does not use the data available? Would our interest be maintained at a higher pitch if all facts had to be discovered, and then used or rejected deliberately? Do we get our maximum of satisfaction from the act of relating together the elements present?

The problem given here would be an "intelligence test" for a ten year old, simply because the only automatic method available to him would be the tedious business of trial and error, whereas he could be "right first time" by marshalling the facts.

From real life data he knows that a father is over 20, say, and under 80. Mathematical facts are the square numbers possible in this range and also that the father's age is an even number both now and four years ago. So we have 36 and 64 as the only possibilities. Moreover, 36 is "more likely," so that the only arithmetic to be done is the check.

Beyond the question of how problems are solved lie the more important ones "what is worth while?" and "is solving problems educational?" If, as it seems to me that we do, we try to reduce them to *types* for which automatic techniques are available, I answer in the negative.

What can we do to improve the situation? Could we collect, invent and analyse problems which do arouse interest, and find a way of presenting them to our pupils so that they learn how to discover in them the facts which will be helpful?

Would someone undertake to collect ideas and examples?

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## GROUP MEETINGS.

### WEST RIDING GROUP.

The Group held a successful meeting in Hyde Park Secondary School, Doncaster, on Saturday, 31st January. Miss M. J. Meetham, of Sheffield Training College, led a discussion on the teaching of area and thought that insufficient time was spent in developing the early concept of area: children should realise that in order to measure surface, the units of linear measurement already known were inadequate. The rules for finding areas of standard shapes should develop from actual experience, well after the concept was firmly established. The discussion centred on the various practical applications of area-finding that were possible in schools.

After lunch, Mr. B. Atkin, of Shirebrook Grammar School, spoke on "Introducing Geometry." He considered that a tradition of blackboard work was largely responsible for the usual introduction to geometry. He proceeded to explain, with the help of his own models, how his own development of the topic commenced with the study of three dimensional solids and led naturally to the formulation of fundamental geometric concepts. The regular solids were used extensively and enabled symmetry to be developed as a powerful mathematical tool.

Details of future meetings are available from the Secretary, Mr. R. W. Shaw, 23 Pipeing Lane, Scawthorpe, Doncaster.

R.W.S.

### NORTH-WESTERN GROUP.

Nearly 300 teachers attended a course on *Secondary School Mathematics—Content and Method* at Manchester on May 9th.

After an introduction by Mr. R. H. Cripwell of Didsbury Training College, the session opened with a bang! 'True Hope is swift,' says Shakespeare, and so he was as he surged into action demanding spot answers from us on the probable shapes to be seen when he cut his folded paper in various places. *My Approach to Teaching Method—Lecture by C. Hope, Esq.*, was on our programme sheet, but the formal title was scant warning of the strong opinions to be expressed. Later in the day, challenges were made against some of these opinions, but the sincerity behind the opinions was always apparent.

'Too many people think that maths. is a thing we were taught to do in a Primary school, so that we could do it in a Secondary School, so that we could do it at College or in a University, so that we can teach it in a Primary or Secondary School, so that pupils can. . . . And round the vicious circle goes.' Mr. Hope said this with feeling, and told us that he was horrified that so many embyro teachers had little or no conception of the real meaning of mathematics. This meaning had to be found, next mathematics had to be learned and understood, and then the

would-be teacher could ply his craft using a notebook and analysing the results of his teaching. Differences in children must be emphasised; groups of like standard must be together. The experiences in learning must be organised by the thinker. 'There must be no brainwashing; we've got to give them the right to be wrong!'

In the lesson on the introduction to algebra taken by Mr. Collins, Cuisenaire rods were used and a language developed from the initials of the different colours. A ripple of interest was felt in the observing teachers when initials were running out and we finished up with brown (B), blue (K for kingfisher blue) and black (C for charcoal). Acknowledging that the atmosphere was artificial, I still feel, as did others, that the use of the rods was pushed rather too much. I am still in doubt as to how I could feasibly show the difference between B + K and B × K or BK. Perhaps we ask too much of these introductory aids?

In the afternoon Mr. Hope laid before us a vista of Secondary Modern Mathematics. He talked about patterns in numbers from which we could find general laws, including $3 \times 3 - 2 \times 1 = 7$, $4 \times 4 - 3 \times 2 = 10$, to discover $n^2 - (n - 1)(n - 2)$. He showed peculiar properties of certain numbers of the type $153 = 1^3 + 5^3 + 3^3$. He pointed out to us the idea of the greatest area obtainable from a given perimeter, shown graphically. 'When in doubt, graph it out' we understood to be the current aphorism of Mr. Hope's students.

The day's events concluded with a healthy clash of opinions on the platform as to the choice of example in the early stages of trigonometry. From a practising and practical point of view, Mr. Birtwistle had the sound point that the ratios used were to fit the actual angles measured, say to the nearest degree, rather than that easy ratios were used which probably represented angles unlikely to be measured. However, to quote again, this time from the Bible, the last word was hurled and we found it was 'Hope to the end!'

J. F. Ellis.

## A.T.A.M. FILM UNIT

Summary of Accounts 25.1.58 to 5.1.59.

| Income | £ | s. | d. | Expenditure | £ | s. | d. |
|---|---|---|---|---|---|---|---|
| Balance in hand, Jan. 1958 .. .. | 18 | 10 | 11 | Purchase and processing film .. .. .. .. | 34 | 7 | 0 |
| Nuffield Foundation .. | 50 | 0 | 0 | Drawing Materials, etc... | 5 | 5 | 6 |
| Donation.. .. .. .. | 6 | 9 | 2 | Balance in hand, Jan. 1959.. .. .. | 35 | 7 | 7 |
| | 75 | 0 | 1 | | 75 | 0 | 1 |

*Signed*, T. J. Fletcher (*Director*)

Summary of full accounts which were examined and found correct by I. Harris and D. P. Prichard.

## BOOK REVIEWS.

### From Actions to Operations.

C. Gattegno. Cuisenaire Co., pp 56. 5s. 0d.

This pamphlet arises out of some lessons given to deaf children and is mainly concerned with deriving some mathematics from Cuisenaire material and Geo-boards. He begins by showing how the rods enable him to develop addition and subtraction '*without using counting*' (the italics are his). In fact he develops the conditions that the elements of the set of rods shall form a group *under addition and uses the symbols 1, 2, . . . .10 as names for the members of the quotient set. In practice, in my experience it is very difficult to avoid ordinality obtained by ranging rods in order — in fact a procedure so much like counting as to be synonymous with it. The first chapters develop the use of rods to illustrate the familiar processes of arithmetic as far as percentages, and Dr. Gattegno fails to make clear how at some stages the rods are entities being described by the coventional symbols and yet at others are the symbols by which the conventional thought processes are developed. The usual legerdemain for division of fractions illustrates this ambiguity of language and of meaning. There is involved here a fundamental principle: should we dictate to the children the mathematics we know (teach them to cancel when multiplying by dropping a rod) or develop the mathematics the children use by the methods they employ, discussing alternative methods in the light of the child's understanding?

The second half of the book is concerned with Geometry. It is a pity that Dr. Gattegno's anxiety puts him in such false positions. The Geo-boards is as much physical experience as an experiment to determine 'g'. It is essentially an experimental method aimed at the transition to Stage B Geometry. Dr. Gattegno makes the claim that he 'will not attempt to introduce geometry through physical experiments. . .' This chapter will be found very useful to those who are embarking on the usual (Euclidean) geometry syllabus and wish to make use of Geo-boards. It is a pity that Dr. Gattegno did not take this opportunity of showing the use of Geo-boards in (Galois) finite geometries and the consequent possibilities of beginning 20th Century Geometry with more gifted children.

A short section on teaching algebra via Cuisenaire rods follows but makes no mention of the possibilities of the three dimensional nature of the material in developing a study of the cubic function. The Cuisenaire rods form a finite set: why has no-one made use of this fact in using them to develop the theory of numbers (congruences), Diophontine equations ($ax + by = c$) for which they are better fitted than developing ideas of continuity?

The pamphlet concludes with an appendix in which Dr. Gattegno summarises his experiences. This booklet will be found to be a very useful adjunct to *Numbers in colour* but not nearly so stimulating. C.H.

* The set of rods do not form a group without further conventions.

### Examples in Applied Mathematics.

F. W. Kellaway. Blackie (Second Edition) pp vi+127. 7s. 6d.

This edition is a 'complete revision' of the original book together with additional questions to make the number of problems greater than 500. Like its predecessor, this collection is suitable for 'A' level G.C.E. Approximately half the book is devoted

to Statics including problems in elasticity and bending moments, a third is devoted to dynamics, and the remainder to hydrostatics.

Many of the examples in Statics are readily 'mocked up' with elementary apparatus. All the traditional examination types are there. In Dynamics there is a good selection of examples on variable acceleration and the usual related graphical questions. I would have liked to have seen more examples on this topic. The usual questions on Newton's Laws, pulleys, inclined planes, relative velocity, projectiles, etc., follow. There is a wide range of applications, more than sufficient to guarantee familiarity with the majority of situations devised by 'A' level examiners. The teacher who is accustomed to discuss, compile notes and then set examples will find that this collection is not quite sufficiently well graded to meet his needs; there are insufficient elementary examples to set a pattern and to give confidence in applying a principle.

Nevertheless, any teacher will find the book an invaluable addition to the individual scholar's library of books; a ready source of additional examples and a good varied collection for learning the skills and techniques required of examination candidates. C.H.

AN INTRODUCTION TO THE THEORY OF GROUPS.

P. S. Alexandroff. Blackie & Sons. pp. 112. 17s. 6d.

One is constantly being exhorted nowadays to include some 'Modern Mathematics' in the pre-degree courses in schools and colleges, but many teachers find the shortage of suitable books somewhat frustrating. This book is a useful contribution to this need.

It is not a book for everyman: the gifted pupil will find the book stimulating; the sophisticated reader will find a good introduction to an interesting field of study. The vocabulary, concepts and principles of the theory of groups are adequately introduced and illustrated by groups of permutations and groups of rotations leading up to Cayley's important theorem on the isomorphism of every finite group with a particular group of permutations. The book continues with cyclic subgroups, simple groups of movements, congruence groups and rotation groups of pyramid, tetrahedron, cube, octahedron and icosahedron. The book concludes with chapters on invariant sub-groups and homomorphic mappings.

Each section has simple exercises which enable a reader to test his understanding of what has gone before. The exercises are well within the competence of a 'Scholarship' boy, but perhaps lack examples which could be taken further. There is a short bibliography: it is a pity that only English texts are given and that no reference is made to the cheap Dover Editions of Kinosh and Carmichael.

This book is useful to the would-be professional mathematician. It is a pity that illustrations which showed the power of the theory in meeting difficulties in other branches of mathematics are not given. To a novitiate the theory would appear rather precious, an elaborate symbolism for turning a tetrahedron round. The Training Colleges will find this lack of background rather confounds the would-be teacher but nevertheless they will find it a very useful addition to the library shelves: a book to read after the spate of paragraphs on 'Groups' in the popular accounts of mathematics and a book to preface Klein's Icosahedron for the student who wishes to make a special study in this field. C.H.

## University Mathematics.

J. Blakey. Blackie & Son. (Second Edition) 1958. pp. 581. 35s.

The business of this book is to answer the problems set in the general degree examinations and the author applies himself to the task with economy, administrative skill and exhaustive classification. Nothing detracts from this aim. The essence of the treatment is to prove nothing which may be assumed, a tenet which is observed by most scientists. The result is the purest of impure mathematics: a wonderfully complete reference book of techniques and rough proofs of all the important results and problems commonly encountered in examination papers.

For an examination candidate, any departure from the line leading to the objective can be very irritating and yet one cannot escape the view that examinations ought not to determine the breadth of education even if they *do* measure what they do. One would like to see some allusion to the background of the many topics studied: why are they important mathematically and what place do they occupy in the gamut of knowledge? What exciting results can they lead to? For example, why introduce polar co-ordinates without some reference to orbital mechanics? Why differential equations without some background of their use as scientific descriptions of observed phenomena? Again, why is no doubt raised as to the admissibility of differentiating or integrating infinite series? Why does the author differentiate functions of a complex variable without considering what is meant by $\delta z \longrightarrow 0$, etc.? Why partial differentiation without double integration? This spate of rhetorical questions might be construed as a criticism of general degree examinations. In part it is, but it is also a criticism of the literature available to the examinee, of the view that the piece of paper conferring the degree is more significant than the knowledge and wisdom this paper purports to guarantee.

For the student whose immediate purpose is to pass an examination this book will be excellent. It begins with a rough treatment of convergency, continues through differentiation, integration, partial differentiation, the general conic, polar co-ordinates and the general conicoid to differential equations and spherical trigonometry. It contains no pure geometry. In fact it is based on the old regulations for the London University general degree and covers every type of question which has been set. As a reference in a school library for problems and methods of solution, this volume will be found most useful and scholarship candidates may find it particularly valuable in this connection. There is no discussion of topics, no attempt to prepare for a more rigorous treatment in the light of modern arrangements of the field of mathematics.

Still, as the recent report on Secondary Mathematics has said, 'The Engineering student is concerned only with method and rightly so.' Many a technical college has most certainly adopted this book with profit to their students and honour to themselves and they will continue to do so. Of its type, it is outstandingly good: good in style, in presentation, and in its comprehensive treatment of what it sets out to do. C.H.

## Ordinary Level Algebra Part 1.

T. H. Ward Hill. George G. Harrap & Co. pp. 144.

This is the first of two volumes which are designed to cover the G.C.E. Ordinary Level syllabus and it includes most of the work done in the first two years of a traditional grammar school course. The book is very well produced. Care has obviously been taken with the selection and variation of type, the pages are well-planned and are easy to read. The style of the explanatory matter is simple, and evidently intended to be compehensible to the pupil reading on his own. Except for some loose use of words (e.g. 'If everything subtracts out the answer is 0') this is largely achieved.

But the chief interest to a reviewer is whether a new book says something new, or presents at least part of its material in a better way than he has seen before. Unfortunately, the present volume can be acquitted on both counts. It is difficult to find any section of the book which evidences an attempt to think out the treatment afresh. The introduction to the subject is based on formulae, algebraic notation is introduced without reference to the need for it; the solution of simple equations is reduced to rule immediately; there is no implicit awareness of the operational nature of directed numbers.

These may not seem very damning criticisms. Many elementary algebra text-books use the same approach. Perhaps this is the frightening thing — that a demand still exists for books written from this standpoint. Essentially, a condemnation of this book, and all the others like it, is that it ignores the fact that children think algebraically long before they need to be conscious of algebra as a name and as a collection of rules of procedure. By not using this ability, which all children possess to some degree, by treating it as an irrelevance in the classroom, we are killing the spontaneous interest and enthusiasm which is only waiting to be aroused, guided and employed.

D.H.W.

## Exercises in Elementary Mathematics, Book 1.

K. B. Swaine. Harrap. pp. 186. Price 5s. 6d.

This is a new edition of the book published in 1949 and provides a comprehensive set of exercises designed to meet the requirements of the Alternative Syllabus. It should prove of great value as a supplementary book in the first year of a G.C.E. course and could be profitably used in perhaps the first two years of a Modern School. As is to be expected there is a considerable amount of manipulative drill, but where suitable the examples have been chosen to have a bearing on real life. In particular the section on Statistical Graphs is very good.

Throughout, the effort is made to maintain the unity of Mathematics; for example, the exercises on the manipulation of fraction include algebraic symbols with examples on number. Very little bookwork is included, with few worked examples; the intention being to provide enough to 'jog the memory.' As with most text-books from this publisher the layout and production is of a high and pleasing standard.

D.T.M.

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3rd October, Yeovil.
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(Details in County Circuiar of Courses).

7th November, Wykeham School, N.W. 10.
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18th/20th March, Blackpool: Weekend Conference.
(Details: Mr. C. Birtwistle, 1 Meredith Street, Nelson, Lancs.)

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